

The multi-time correlation functions, free white noise, and the generalized Poisson statistics in the low density limit

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Abstract. In the present paper the low density limit of the non-chronological multitime correlation functions of boson number type operators is investigated. We prove that the limiting truncated non-chronological correlation can be computed using only a sub-class of diagrams associated to non-crossing pair partitions and thus coincide with the non-truncated correlation functions of suitable free number operators. The independent in the limit subalgebras are found and the limiting statistics is investigated. In particular, it is found that the cumulants of certain elements coincide in the limit with the cumulants of the Poisson distribution. An explicit representation of the limiting correlation functions and thus of the limiting algebra is constructed in a special case through suitably defined quantum white noise operators.

1. INTRODUCTION

The reduced dynamics of a quantum open system interacting with a reservoir in certain physical regimes is approximated by Markovian master equations. These regimes include the weak system–reservoir interactions and dilute reservoirs and in the theoretical framework they are described by certain limits. For a weakly interacting system one considers the limit as the coupling constant goes to zero (Weak Coupling Limit, WCL) whereas for a dilute reservoir one considers the limit as the density of the reservoir goes to zero (Low Density Limit, LDL) and an appropriate time rescaling should be performed in order to get a non-trivial limit. The Markovian reduced dynamics in these limits is considered in the review papers by Spohn and Lebowitz^{1,2}. The reduced dynamics in the LDL was considered in details later by Dümcke³ using the method based on the quantum Bogoliubov–Born–Green–Kirkwood–Yvon hierarchy.

The total dynamics in these limits is governed by various quantum stochastic equations. There is a unique up to now approach, called the *stochastic limit method*, which allows an efficient derivation of the stochastic equations in the WCL. This approach is based on the quantum white noise technique and was developed by Accardi, Lu, and Volovich⁴.

The convergence of the evolution operator of the total system in the LDL to a solution of a quantum stochastic equation was proved by Accardi and Lu⁵ and by Rudnicki, Alicki, and Sadowski⁶. Recently the low density limit was investigated with the quantum white noise technique^{7,8}. This technique, well developed for the WCL, was non-trivially modified to include the LDL and for this case was called *the stochastic golden rule for the low density limit*. This technique was applied to the derivation of the quantum stochastic equations in the LDL. An advantage of the obtained equations is that they, in contrast with the exact Schrödinger equation, are explicitly solvable. At the same time they provide a good approximation of the exact dynamics.

The approach of^{7,8} uses the so called Fock-antiFock representation for the canonical commutation relations (CCR) algebra (this representation is unitary equivalent to the Gel'fand–Naimark–Segal representation). The difficulty with this approach is that the creation and annihilation operators in the Fock-antiFock Hilbert space do not describe creation and annihilation of physical particles and thus do not have direct physical meaning. To avoid this difficulty the investigation of the LDL directly in terms of the physical fields was performed⁹. Using this approach the chronological correlation functions in the LDL were found and the corresponding stochastic equations derived.

In the present paper we investigate the low density limit of the non-chronologically ordered correlation functions of boson number type operators. The investigation is related with *ab initio* derivations of quantum stochastic equations describing quantum dynamics of a test particle interacting with a dilute gas. We find the limiting truncated correlation functions of the number type operators and show that they can be computed by representing the number operators through creation and annihilation operators and then considering only a sub-class of diagrams associated to non-crossing pair partitions.

This fact allows to represent the limiting truncated correlation functions as the non-truncated correlation functions of number operators of a free quantum white noise thus making a connection with the Voiculescu free probability theory. We find the limiting statistics and show that the cumulants of certain elements coincide in the limit with the cumulants of the Poisson distribution.

The free probability theory was developed by Voiculescu around 1985 as a way to deal with von Neumann algebras of free groups. Then the theory was separated from this special context and began to develop as an independent field. In particular, applications of the free independence theory to random matrices were found. The details of free probability theory and its applications to random matrices could be found, for example, in references^{10,11}.

Expectations of free random variables are characterized by diagrams associated to non-crossing pair partitions. The vanishing of crossing diagrams in the stochastic weak coupling limit for nonrelativistic QED and for the Anderson model was found in Refs⁴ and¹², respectively, thus making a connection between the WCL and free probability. The WCL is typically described by the quantum Boltzmann statistics⁴. In Ref¹³ a generalized version of Boltzmann commutation relations, the so called entangled commutation relations, was found in the weak coupling limit for nonlinear interactions and possible applications to photon splitting cascades were discussed.

The investigation of the multitime non-chronologically ordered correlation functions could have a connection with the behavior of fluctuations in certain asymptotic regimes. The latter is described in the review paper by Andries, Benatti, De Cock and Fannes¹⁴. In that approach the limiting statistics is defined in terms of ground state distribution determined by non trivial pair partitions. The authors conjecture the appearance of exotic statistics in certain asymptotic regimes. The asymptotic fluctuations are the limiting correlation functions of appropriate centered elements and thus the results of the present paper could be applied to study the fluctuations in the low density limit.

In Sec. II the truncated non-chronologically ordered correlation functions are defined and their low density limit is established (Theorem 1). In Sec. III the irreducible diagrams (pair partitions) which contribute to the limiting correlation functions are found (Theorem 2). In Sec. IV the limiting truncated correlation functions are represented as correlation functions of a suitable free white noise. In Sec. V we identify the independent in the limit subalgebras (Theorem 4) and calculate the limiting cumulants which for some elements coincide with the cumulants of the Poisson distribution (Theorem 5). In Sec. VI an explicit representation of the limiting correlation functions and thus of the limiting algebra is constructed for a special case by using suitable quantum white noise operators.

2. THE CORRELATION FUNCTIONS IN THE LDL

We begin this section with construction of a general class of non-commutative probability spaces relevant for the investigation of the low density limit. The framework of a *-

probability space is used. A relation between the objects defined in this section and the model of a test particle interacting with a dilute gas is given in Appendix A.

Definition 1 A **-probability space* is a pair (\mathcal{A}, ω) , where \mathcal{A} is a unital *-algebra over \mathbb{C} and $\omega : \mathcal{A} \rightarrow \mathbb{C}$ is a state, i.e., a linear normalized, $\omega(1_{\mathcal{A}}) = 1$, and strictly positive functional.

Let \mathcal{H} be a Hilbert space with inner product denoted by $\langle \cdot, \cdot \rangle$ (called as one particle Hilbert space), $\{S_t\}_{t \in \mathbb{R}}$ a one parameter unitary group in \mathcal{H} (a one particle free evolution), \hat{n} a bounded positive operator in \mathcal{H} (density operator) such that $\forall t \in \mathbb{R}$, $S_{-t}\hat{n}S_t = \hat{n}$, and B a countable set of real numbers.

Let $\Gamma(\mathcal{H})$ be the symmetric Fock space over \mathcal{H} . For any trace class self-adjoint operator T acting in \mathcal{H} we denote by $N(T) \equiv d\Gamma(T)$ its second quantization operator in $\Gamma(\mathcal{H})$ and extend this definition by complex linearity to the set of all trace class operators $\mathcal{T}(\mathcal{H})$. For any $T \in \mathcal{T}(\mathcal{H})$, $\omega \in B$, and a positive number $\varepsilon > 0$ we define the following operator in $\Gamma(\mathcal{H})$:

$$N_{T, \omega, \varepsilon}(t) := \frac{e^{-it\omega/\varepsilon}}{\varepsilon} N(S_{t/\varepsilon} T S_{-t/\varepsilon}) \quad (1)$$

Let $L(\mathbb{R}) = \bigcap_{p \in \mathbb{N}} L^p(\mathbb{R})$, where $L^p(\mathbb{R})$ is the space of p -power integrable functions over \mathbb{R} . For any open subset $\Lambda \subseteq \mathbb{R}$ let $L(\Lambda)$ be the set of functions from $L(\mathbb{R})$ with support in Λ . We denote by $\mathcal{A}_{\Lambda, \varepsilon}$ the *-algebra generated by operators $N_{T, \omega, \varepsilon}(\varphi) := \int dt \varphi(t) N_{T, \omega, \varepsilon}(t)$ with $T \in \mathcal{T}(\mathcal{H})$, $\omega \in B$, $\varphi \in S(\Lambda)$ and denote $\mathcal{A}_{\varepsilon} := \mathcal{A}_{\mathbb{R}, \varepsilon}$.

Let $A^{\pm}(g)$, $g \in \mathcal{H}$ be the creation and annihilation operators in $\Gamma(\mathcal{H})$ [we denote in the sequel $A^{-}(g) \equiv A(g)$] with the canonical commutation relations $[A(f), A^{+}(g)] = \langle f, g \rangle$ and let \mathcal{A}_{CCR} be the algebra of polynomials in $A^{\pm}(\cdot)$. Any operator $N(T)$ can be represented in terms of the creation and annihilation operators. For example, if $T = |f\rangle\langle g|$, where we use Dirac's notations for elements $f, g \in \mathcal{H}$, then $N(T) = A^{+}(f)A(g)$. An arbitrary operator $N(T)$ can be expressed in terms of A^{\pm} using the fact that any trace class operator T is a limit of finite rank operators. Thus the algebra $\mathcal{A}_{\varepsilon}$ is a subalgebra of \mathcal{A}_{CCR} .

Let $\omega_{\hat{n}}$ be a gaussian gauge-invariant mean-zero state on \mathcal{A}_{CCR} with the two point correlation function $\omega_{\hat{n}}(A^{+}(f)A(g)) := \langle g, \hat{n}f \rangle$ (thus $\omega_{\hat{n}}(N(T)) = \text{Tr}(\hat{n}T)$ and here we use the assumption for T being trace class). Denoting by the same symbol its restriction to $\mathcal{A}_{\Lambda, \varepsilon}$, we finally have for any $\varepsilon > 0$ and for any open subset $\Lambda \subseteq \mathbb{R}$ the *-probability space $(\mathcal{A}_{\Lambda, \varepsilon}, \omega_{\varepsilon\hat{n}})$.

Remark 1 The condition $\forall t: S_{-t}\hat{n}S_t = \hat{n}$ leads to the invariance of the state $\omega_{\hat{n}}$ under the free evolution generated by S_t .

With the notations above we define the non-chronologically ordered multitime correlation functions as

$$W_{\varepsilon, \hat{n}, T_1, \omega_1, \dots, T_n, \omega_n}(t_1, \dots, t_n) := \omega_{\varepsilon\hat{n}}(N_{T_1, \omega_1, \varepsilon}(t_1) \dots N_{T_n, \omega_n, \varepsilon}(t_n)) \quad (2)$$

$$W_{\varepsilon, \hat{n}, T_1, \omega_1, \dots, T_n, \omega_n}(\varphi_1, \dots, \varphi_n) := \omega_{\varepsilon\hat{n}}(N_{T_1, \omega_1, \varepsilon}(\varphi_1) \dots N_{T_n, \omega_n, \varepsilon}(\varphi_n)) \quad (3)$$

We will use for the correlation functions (2) and (3) also the shorter notations $W_\varepsilon(t_1, \dots, t_n)$ and $W_\varepsilon(\varphi_1, \dots, \varphi_n)$. The reason for introducing the averaged operators $N_{T, \omega, \varepsilon}(\varphi)$ and the averaged correlation functions (3) is that, as we will show below, the non-averaged operators $N_{T, \omega, \varepsilon}(t)$ and the correlation functions (2) in the limit as $\varepsilon \rightarrow 0$ become singular distributions. Clearly, one has the relation

$$W_\varepsilon(\varphi_1, \dots, \varphi_n) = \int dt_1 \dots dt_n W_\varepsilon(t_1, \dots, t_n) \varphi_1(t_1) \dots \varphi_n(t_n)$$

Definition 2 *The truncated correlation functions $W_\varepsilon^T(t_1, \dots, t_n)$ are defined for $n = 1$ by $W_\varepsilon^T(t_1) := W_\varepsilon(t_1)$ and for $n > 1$ by induction through the relation:*

$$W_\varepsilon(t_1, \dots, t_n) = W_\varepsilon^T(t_1, \dots, t_n) + \sum_{l=2}^n \sum' W_\varepsilon^T(t_{i_1}, \dots, t_{i_{k_1}}) \\ \times W_\varepsilon^T(t_{i_{k_1+1}}, \dots, t_{i_{k_2}}) \dots W_\varepsilon^T(t_{i_{k_l}}, \dots, t_{i_n})$$

where \sum' is the sum over $i_1 < i_2 < \dots < i_{k_1}$, $i_{k_1+1} < \dots < i_{k_2}$, \dots , $i_{k_l+1} < \dots < i_n$.

The truncated correlation functions are often used in quantum field theory and in quantum kinetic theory¹⁵. They entirely determine the corresponding non-chronological correlation functions. Thus the investigation of the limit of the non-chronological correlation functions can be reduced to the investigation of the limit of the truncated correlation functions.

We define the "projection" $P_E := (2\pi)^{-1} \int dt S_t e^{-itE}$ [it has the property $P_E P_{E'} = \delta(E - E') P_E$] and for any $k = 1, 2, \dots, n$ denote $\tilde{\omega}_k = \omega_n + \dots + \omega_k$. The following theorem states the low density limit of the truncated correlation functions.

Theorem 1 *One has the limit in the sense of distributions in variables t_1, \dots, t_n :*

$$\lim_{\varepsilon \rightarrow 0} W_{\varepsilon, \hat{n}, T_1, \omega_1, \dots, T_n, \omega_n}^T(t_1, \dots, t_n) = (2\pi)^{n-1} \delta(t_2 - t_1) \dots \delta(t_n - t_{n-1}) \\ \times \delta_{\tilde{\omega}_1, 0} \int dE \text{Tr} \left[\hat{n} P_{E+\tilde{\omega}_1} T_1 P_{E+\tilde{\omega}_2} T_2 \dots P_{E+\tilde{\omega}_n} T_n \right] \quad (4)$$

where Tr denotes trace and $\delta_{\tilde{\omega}_1, 0}$ is the Kronecker delta symbol.

The theorem is a corollary of Theorem 2 from Section 3.

3. THE NON-TRIVIAL DIAGRAMS

In the present section we investigate the low density limit of the non-chronologically ordered correlation functions for the particular case of operators of the form $T_l = |f_l\rangle\langle g_l|$ and find the diagrams which are non-trivial in the low density limit.

In order to simplify the notations we will use the following **energy representation** for the creation and annihilation operators:

$$A_l^+ := \frac{e^{it_l E_l / \varepsilon}}{\sqrt{\varepsilon}} A^+(P_{E_l} f_l); \quad A_l := \frac{1}{\sqrt{\varepsilon}} A(S_{t_l / \varepsilon} g_l)$$

(a slightly different version of the energy representation was introduced in⁷). One has $N_{T_l, \omega_l, \varepsilon}(t_l) = e^{-it_l \omega_l / \varepsilon} \int dE_l A_l^+ A_l$. Notice that the operator A_l^+ is not the adjoint of A_l . The symbols A_l , A_l^+ are used only to simplify the notations below.

A multitime correlation function can be expressed using Gaussianity of the state $\omega_{\hat{n}}$ and the energy representation for the creation and annihilation operators as

$$W_{\varepsilon, \hat{n}, T_1, \omega_1, \dots, T_n, \omega_n}(t_1, \dots, t_n) = \exp\left(-i \sum_{l=1}^n \omega_l t_l / \varepsilon\right) \sum' \int dE_1 \dots dE_n \omega_{\varepsilon \hat{n}}(A_{i_1}^+ A_{j_1}) \dots \times \omega_{\varepsilon \hat{n}}(A_{i_k}^+ A_{j_k}) \omega_{\varepsilon \hat{n}}(A_{j_{k+1}} A_{i_{k+1}}^+) \dots \omega_{\varepsilon \hat{n}}(A_{j_n} A_{i_n}^+) \quad (5)$$

where \sum' is the sum over $k = 1, \dots, n$, $1 = i_1 < i_2 < \dots < i_k$, $j_{k+1} < \dots < j_n$, $i_l \leq j_l$ for $l = 1, \dots, k$ and $j_l < i_l$ for $l = k+1, \dots, n$. The sum contains terms of the form

$$\omega_{\varepsilon \hat{n}}(A_{i_1}^+ A_{j_1}) \dots \omega_{\varepsilon \hat{n}}(A_{i_k}^+ A_{j_k}) \omega_{\varepsilon \hat{n}}(A_{j_{k+1}} A_{i_{k+1}}^+) \dots \omega_{\varepsilon \hat{n}}(A_{j_n} A_{i_n}^+) \quad (6)$$

To each such term we associate a diagram by pairing in the string $A_1^+ A_1 A_2^+ A_2 \dots A_n^+ A_n$ the operators $A_{i_l}^+$ and A_{j_l} for $l = 1, 2, \dots, n$.

Definition 3 We say that the expression (6) corresponds to a reducible diagram if there exists a nonempty subset $I \subset \{1, \dots, n\}$ (strict inclusion) such that $i_l \in I \Leftrightarrow j_l \in I$. Otherwise we say that the expression (6) corresponds to an irreducible diagram.

An important property of the truncated correlation functions (Def. 2) is that they keep only all irreducible diagrams. The following are the examples of irreducible (first) and reducible (second) diagrams for $n = 2$:

$$\begin{array}{cc} \overline{\boxed{}} & \boxed{} \boxed{} \\ A_1^+ A_1 A_2^+ A_2 & A_1^+ A_1 A_2^+ A_2 \end{array} \quad (7)$$

Given an reducible diagram, one can represent the set $\{1, \dots, n\}$ as a union of several disjoint subsets I_1, \dots, I_l such that the diagram contains only pairings between operators with indices from the same subsets. In this sense a general reducible diagram can be represented as a union of mutually disjoint irreducible diagrams. Examples of the truncated correlation functions, the corresponding irreducible diagrams, and their limits as $\varepsilon \rightarrow 0$ for $n = 1, 2, 3$ are given below.

Example 1 $n = 1$. The invariance of the state under the free evolution leads to the identity $W_\varepsilon^T(t) \equiv W_\varepsilon(t) \equiv W_\varepsilon(0) = \langle g_1, \hat{n} f_1 \rangle$.

Example 2 $n = 2$. One has

$$\begin{aligned} W_\varepsilon^T(t_1, t_2) &= W_\varepsilon(t_1, t_2) - W_\varepsilon(t_1) W_\varepsilon(t_2) = \int dE_1 dE_2 \omega_{\varepsilon \hat{n}}(A_1^+ A_2) \omega_{\varepsilon \hat{n}}(A_1 A_2^+) \\ &= \int dE_1 dE_2 \frac{e^{i(t_2 - t_1)(E_2 - E_1)/\varepsilon}}{\varepsilon} \langle g_2, P_{E_1} \hat{n} f_1 \rangle \langle g_1, (1 + \varepsilon \hat{n}) P_{E_2} f_2 \rangle \end{aligned} \quad (8)$$

This expression corresponds to the first (irreducible) diagram in (7) which is non-zero in the limit. Application of Lemma 1 (see Appendix B) to the r.h.s. of (8) gives

$$\lim_{\varepsilon \rightarrow 0} W_\varepsilon^T(t_1, t_2) = 2\pi \delta(t_2 - t_1) \int dE \langle g_2, P_E \hat{n} f_1 \rangle \langle g_1, P_E f_2 \rangle$$

Example 3 $n = 3$. One has

$$W_\varepsilon^T(t_1, t_2, t_3) = \int dE_1 dE_2 dE_3 \left[\omega_{\varepsilon\hat{n}}(A_1^+ A_3) \omega_{\varepsilon\hat{n}}(A_1 A_2^+) \omega_{\varepsilon\hat{n}}(A_2 A_3^+) \right. \\ \left. + \omega_{\varepsilon\hat{n}}(A_1^+ A_2) \omega_{\varepsilon\hat{n}}(A_1 A_3^+) \omega_{\varepsilon\hat{n}}(A_2^+ A_3) \right]$$

This expression corresponds to the sum of the two irreducible diagrams:

$$\overline{\begin{array}{c} \square \quad \square \quad \square \\ A_1^+ A_1 A_2^+ A_2 A_3^+ A_3 \end{array}} + \overline{\begin{array}{c} \square \quad \square \quad \square \\ A_1^+ A_1 A_2^+ A_2 A_3^+ A_3 \end{array}}$$

In this case only the first diagram is non-zero in the limit and Lemma 1 gives

$$\lim_{\varepsilon \rightarrow 0} W_\varepsilon^T(t_1, t_2, t_3) = (2\pi)^2 \delta(t_3 - t_2) \delta(t_2 - t_1) \int dE \langle g_3, P_E \hat{n} f_1 \rangle \langle g_1, P_E f_2 \rangle \langle g_2, P_E f_3 \rangle$$

The case of arbitrary n is described by the following theorem.

Theorem 2 Let $T_l = |f_l\rangle\langle g_l|$, where $f_l, g_l \in \mathcal{H}$ for $l = 1, 2, \dots, n$. One has the limit in the sense of distributions in variables t_1, \dots, t_n :

$$\lim_{\varepsilon \rightarrow 0} W_{\varepsilon, \hat{n}, T_1, \omega_1, \dots, T_n, \omega_n}^T(t_1, \dots, t_n) = (2\pi)^{n-1} \delta(t_2 - t_1) \dots \delta(t_n - t_{n-1}) \\ \times \delta_{\tilde{\omega}_1, 0} \int dE \langle g_n, P_E \hat{n} f_1 \rangle \langle g_1, P_{E+\tilde{\omega}_2} f_2 \rangle \dots \langle g_{n-1}, P_{E+\tilde{\omega}_n} f_n \rangle \quad (9)$$

For each n only the following irreducible diagram is non-zero as $\varepsilon \rightarrow 0$:

$$\overline{\begin{array}{c} \square \quad \square \quad \square \quad \square \quad \square \quad \square \\ A_1^+ A_1 A_2^+ A_2 A_3^+ A_3 A_4^+ \dots A_{n-1} A_{n-1}^+ A_n \end{array}} \quad (10)$$

Proof. Case (a): $\omega_1 = \omega_2 = \dots = \omega_n = 0$. Using the correlation functions

$$\omega_{\varepsilon\hat{n}}(A_{i_\alpha}^+ A_{j_\alpha}) = e^{i(t_{i_\alpha} - t_{j_\alpha})E_{i_\alpha}/\varepsilon} \langle g_{j_\alpha}, \hat{n} f_{i_\alpha} \rangle \\ \omega_{\varepsilon\hat{n}}(A_{j_\beta} A_{i_\beta}^+) = \frac{e^{i(t_{i_\beta} - t_{j_\beta})E_{i_\beta}/\varepsilon}}{\varepsilon} \langle g_{j_\beta}, (1 + \varepsilon\hat{n}) f_{i_\beta} \rangle$$

one can write (6) as

$$\frac{1}{\varepsilon^n} \exp \left\{ i \left[(t_1 - t_{j_1}) E_1 + \dots + (t_{i_n} - t_{j_n}) E_{i_n} \right] / \varepsilon \right\} \left(\varepsilon^k F(E) + O(\varepsilon^{k+1}) \right) \quad (11)$$

where

$$F(E) = \prod_{l=1}^k \langle g_{j_l}, P_{E_l} \hat{n} f_{i_l} \rangle \prod_{l=k+1}^n \langle g_{j_l}, P_{E_{i_l}} f_{i_l} \rangle$$

Define the permutations p_i and p_j of the set $(1, \dots, n)$ by $p_i(l) = i_l$ and $p_j(l) = j_l$ for $l = 1, \dots, n$ and let $p_\alpha = p_i p_j^{-1}$. Consider the expression in the square brackets in the exponent in (11). The term proportional to t_l in this expression has the form $t_l(E_l - E_{\alpha_l})$, where $\alpha_l = p_\alpha(l)$. Thus (11) can be written as

$$\frac{1}{\varepsilon^n} \exp \left\{ i \left[t_n(E_n - E_{\alpha_n}) + \dots + t_1(E_1 - E_{\alpha_1}) \right] / \varepsilon \right\} \left(\varepsilon^k F(E) + O(\varepsilon^{k+1}) \right)$$

and with the notations $\Omega_l(E) = E_n + \dots + E_l - E_{\alpha_n} - \dots - E_{\alpha_l}$ for $l = 2, \dots, n$ as

$$\frac{e^{i(t_n - t_{n-1})\Omega_n(E)/\varepsilon}}{\varepsilon} \dots \frac{e^{i(t_2 - t_1)\Omega_2(E)/\varepsilon}}{\varepsilon} \left(\varepsilon^{k-1} F(E) + O(\varepsilon^k) \right) \quad (12)$$

If the expression (6) corresponds to an irreducible diagram then the functions $\Omega_l(E)$ are linearly independent and, since they are linear in their arguments, the convolution $\delta(\Omega_2(E)) \dots \delta(\Omega_n(E))$ is well defined.

In the case $k > 1$, since for any $l = 2, \dots, n$ (see Lemma 1):

$$\lim_{\varepsilon \rightarrow 0} \frac{e^{i(t_l - t_{l-1})\Omega_l(E)/\varepsilon}}{\varepsilon} = 2\pi\delta(t_l - t_{l-1})\delta(\Omega_l(E)) \quad (13)$$

and $k - 1 > 0$, the limit of (12) equals to zero.

In the case $k = 1$ the expression (6) corresponds to the diagram (10) and one has

$$\begin{aligned} & \omega_{\varepsilon\hat{n}}(A_1^+ A_n) \omega_{\varepsilon\hat{n}}(A_1 A_2^+) \dots \omega_{\varepsilon\hat{n}}(A_{n-1} A_n^+) \\ &= \frac{e^{i(t_n - t_{n-1})\Omega_n(E)/\varepsilon}}{\varepsilon} \dots \frac{e^{i(t_2 - t_1)\Omega_2(E)/\varepsilon}}{\varepsilon} \left(F(E) + O(\varepsilon) \right) \end{aligned} \quad (14)$$

where $\Omega_l(E) = E_l - E_1$. Using (13) one finds that the limit of the r.h.s. of (14) is

$$\begin{aligned} & (2\pi)^{n-1} \delta(t_2 - t_1) \dots \delta(t_n - t_{n-1}) \delta(E_2 - E_1) \dots \delta(E_n - E_1) \\ & \times \langle g_n, P_{E_1} \hat{n} f_1 \rangle \langle g_1, P_{E_2} f_2 \rangle \dots \langle g_{n-1}, P_{E_n} f_n \rangle \end{aligned}$$

Integration over $E_1 \dots E_n$ gives the equality (9) in the case (a).

Case (b): arbitrary $\omega_1, \dots, \omega_n$. In this case the expression (14) in the decomposition (5) is multiplied by the factor $\exp(-i \sum_l \omega_l t_l / \varepsilon)$. The product can be written as

$$\frac{e^{i(t_n - t_{n-1})(\Omega_n(E) - \tilde{\omega}_n)/\varepsilon}}{\varepsilon} \dots \frac{e^{i(t_2 - t_1)(\Omega_2(E) - \tilde{\omega}_2)/\varepsilon}}{\varepsilon} e^{-it_1 \tilde{\omega}_1 / \varepsilon} \left(F(E) + O(\varepsilon) \right)$$

If $\tilde{\omega}_1 = 0$ then the statement of the theorem follows by the same arguments as in the case (a). If $\tilde{\omega}_1 \neq 0$ then the limit of this term equals to zero by Riemann-Lebesgue lemma due to the presence of the rapidly oscillating factor $\exp(-it_1 \tilde{\omega}_1 / \varepsilon)$. \square

4. THE FREE WHITE NOISE NUMBER OPERATORS

In the present section we show that the limiting truncated correlation functions coincide with the complete (i.e., non-truncated) correlation functions of the free white noise number operators.

Definition 4 *Free white noise operators* $N_T(t)$ are the operators satisfying the multiplication rule

$$N_T(t) N_{T'}(t') = \delta(t - t') N_{T * T'}(t) \quad (15)$$

where the $*$ -product of any two operators T and T' is defined by $T * T' := 2\pi \int dE P_E T P_E T'$.

Remark 2 We call the operators $N_T(t)$ as **free (or Boltzmann) number operators** since they can be constructed using the creation and annihilation operators $B_f^\pm(t)$ satisfying the free relations $B_f^-(t) B_g^+(t') = 2\pi\delta(t - t') \langle f, g \rangle$. In fact, define $N_{|f\rangle\langle g|}(t) := \int dE B_{P_E f}^+(t) B_{P_E g}^-(t)$ and extend this definition by linearity to any T . Then such defined operators satisfy the relation (15).

Let \mathcal{A} be the algebra generated by the free white noise operators $N_T(t)$ and let $\phi_{\hat{n}}$ be the state on \mathcal{A} characterized by $\phi_{\hat{n}}(N_T(t)) = \text{Tr}(\hat{n}T)$.

Theorem 3 *One has the equality*

$$\lim_{\varepsilon \rightarrow 0} W_{\varepsilon, \hat{n}, T_1, 0, \dots, T_n, 0}^T(t_1, \dots, t_n) = \phi_{\hat{n}}(N_{T_1}(t_1) \dots N_{T_n}(t_n)) \quad (16)$$

Proof. By direct calculations using the Eq. (4) and the relation (15).

The existence of the representation of the limiting truncated correlation functions by the free white noise number operators is related to the fact that only a sub-class of the non-crossing irreducible diagrams survives in the low density limit. We emphasize however, that the l.h.s. of Eq. (16) is the limit of a truncated correlation function whereas the r.h.s. contains the complete correlation function.

5. INDEPENDENCE AND THE GENERALIZED POISSON STATISTICS IN THE LDL

The fact that the limiting truncated correlation functions are the distributions in variables t_1, \dots, t_n with support at $t_1 = \dots = t_n$ leads to the appearance of independent subalgebras in the low density limit. In the beginning of this section we remind the basic notions of independent subalgebras and of cumulants. Then we find the asymptotically independent subalgebras of \mathcal{A}_ε and discuss the limiting statistics. We show that the cumulants and the moments of certain elements in the algebra \mathcal{A}_ε in the low density limit coincide with the cumulants and the moments of the Poisson distribution.

Definition 5 *Let (\mathcal{A}, ω) be a $*$ -probability space. A family of unital $*$ -subalgebras $\{\mathcal{A}_i\}_{i \in I}$, $\mathcal{A}_i \subset \mathcal{A}$, is called **independent** if $\omega(a_1 \dots a_n) = 0$ whenever $a_l \in \mathcal{A}_{i_l}$, $\omega(a_l) = 0$, and $k \neq l$ implies $i_k \neq i_l$.*

Definition 6 *Let (\mathcal{A}, ω) be a $*$ -probability space. **Cumulants** of the space (\mathcal{A}, ω) are the multilinear functionals $\kappa_n : \mathcal{A}^n \rightarrow \mathbb{C}$, $n \geq 1$, uniquely determined by $\kappa_1(a) := \omega(a)$, $a \in \mathcal{A}$, and for $n > 1$ by induction through the relation:*

$$\omega(a_1 \dots a_n) = \sum_{\pi, \pi = \{A_1, \dots, A_k\}} \kappa_{|A_j|}((a_1, \dots, a_n)|A_j)$$

where the sum is over all partitions π of the set $\{1, \dots, n\}$ and $”(a_1, \dots, a_n)|A”$ designates the set of a_i with $i \in A$.

Remark 3 *The cumulants $\kappa_n^{(\varepsilon)}$ for a $*$ -probability space $(\mathcal{A}_\varepsilon, \omega_{\varepsilon \hat{n}})$ are directly related to the truncated correlation functions. Namely, if $a_1 = N_{T_1, \omega_1, \varepsilon}(\varphi_1), \dots, a_n = N_{T_n, \omega_n, \varepsilon}(\varphi_n)$, then $\kappa_n^{(\varepsilon)}(a_1, \dots, a_n) = W_{\varepsilon, \hat{n}, T_1, \omega_1, \dots, T_n, \omega_n}^T(\varphi_1, \dots, \varphi_n)$.*

For the analysis of independence in the low density limit we introduce the notion of asymptotically independent subalgebras for a $*$ -probability space $(\mathcal{A}_\varepsilon, \omega_{\varepsilon \hat{n}})$.

Definition 7 Let $(\mathcal{A}_\varepsilon, \omega_{\varepsilon\hat{n}})$ be a $*$ -probability space for the LDL. We say that a family of subalgebras $\mathcal{A}_{1,\varepsilon}, \dots, \mathcal{A}_{l,\varepsilon}$ of \mathcal{A}_ε is **asymptotically independent** if

$$\lim_{\varepsilon \rightarrow 0} \omega_{\varepsilon\hat{n}}(a_1, \dots, a_n) = 0$$

whenever $a_l \in \mathcal{A}_{i_l,\varepsilon}$, $\omega_{\varepsilon\hat{n}}(a_l) = 0$, and $k \neq l$ implies $i_k \neq i_l$.

The next theorem identifies asymptotically independent subalgebras of \mathcal{A}_ε .

Theorem 4 Let $\Lambda_1, \dots, \Lambda_l$ be a family of disjoint open subsets in \mathbb{R} . Then the family of subalgebras $\mathcal{A}_{\Lambda_1,\varepsilon}, \dots, \mathcal{A}_{\Lambda_l,\varepsilon}$ is asymptotically independent.

The proof follows from the fact that the truncated correlation functions become in the limit as $\varepsilon \rightarrow 0$ distributions in variables t_1, \dots, t_n with support at $t_1 = t_2 = \dots = t_n$. \square

Now let us analyze the statistics which appears in the low density limit. From Theorem 1 and the relation between the cumulants and the truncated correlation functions it follows that l -th cumulant for the element $a = N_{T,\omega,\varepsilon}(\varphi)$ in the limit has the form

$$\kappa_l(a, \dots, a) = \lim_{\varepsilon \rightarrow 0} W_{\varepsilon,\hat{n},T,\omega,\dots,T,\omega}^T(\varphi, \dots, \varphi) = \frac{1}{2\pi} \delta_{\omega,0} \int dt dE \operatorname{Tr} \hat{n} [2\pi \varphi(t) P_E T]^l \quad (17)$$

We specify the further consideration to the case $\mathcal{H} = L^2(\mathbb{R}^3)$. Consider $\hat{n} = 1$ and $S_t = e^{itH_1}$ where H_1 is the multiplication operator by the function $\omega(\mathbf{k}) = |\mathbf{k}|^2$, $\mathbf{k} \in \mathbb{R}^3$. Let T_λ be an integral operator in \mathcal{H} with the kernel $T_\lambda(\mathbf{k}, \mathbf{k}') = (2\pi \sqrt{|\mathbf{k}||\mathbf{k}'|})^{-1} \chi_{[0,\sqrt{\lambda}]}(|\mathbf{k}|) \chi_{[0,\sqrt{\lambda}]}(|\mathbf{k}'|)$, where λ is a positive number and $\chi_{[0,\sqrt{\lambda}]}$ is the characteristic function of the interval $[0, \sqrt{\lambda}]$. Let $\varphi_0(t) = (2\pi)^{-1} \chi_{[0,2\pi]}(t)$.

Theorem 5 Let $a_\lambda = N_{T_\lambda,\omega,\varepsilon}(\varphi_0)$, where T_λ and φ_0 are defined as above. Then for any $l \in \mathbb{N}$ one has

$$\kappa_l(a_\lambda, \dots, a_\lambda) = \lambda \delta_{\omega,0}$$

or equivalently, the cumulants of the element a_λ with $\omega = 0$ coincide in the low density limit with the cumulants of the Poisson distribution with expectation equal to λ .

Proof. The proof of the theorem is based on the direct calculation of the cumulants using Eq. (17). One has

$$\frac{1}{2\pi} \int dt [2\pi \varphi_0(t)]^l = 1$$

One also has

$$\begin{aligned} \int dE \operatorname{Tr} \hat{n} [P_E T_\lambda]^l &= \int dE \int d\mathbf{k}_1 \dots d\mathbf{k}_l \delta(|\mathbf{k}_1|^2 - E) T_\lambda(\mathbf{k}_1, \mathbf{k}_2) \\ &\quad \times \delta(|\mathbf{k}_2|^2 - E) T_\lambda(\mathbf{k}_2, \mathbf{k}_3) \dots \delta(|\mathbf{k}_l|^2 - E) T_\lambda(\mathbf{k}_l, \mathbf{k}_1) \\ &= \int dE \left[T_\lambda(\sqrt{E}, \sqrt{E}) \int d\mathbf{k} \delta(|\mathbf{k}|^2 - E) \right]^l = \int dE \chi_{[0,\sqrt{\lambda}]}(\sqrt{E}) = \lambda. \end{aligned}$$

Thus the r.h.s. of Eq. (17) equals to one. This proves the theorem. \square

Moments of the element a_λ with $\omega = 0$ in the low density limit are equal to the sum over all partitions of the limiting cumulants and given by Touchard polynomials:

$$\lim_{\varepsilon \rightarrow 0} \omega_{\varepsilon \hat{n}}(a_\lambda^n) = \sum_{k=1}^n S(n, k) \lambda^k$$

where $S(n, k)$ is a Stirling number of the second kind, i.e., the number of partitions of a set of size n into k disjoint non-empty subsets. The limiting moments coincide with the moments of the Poisson distribution with expectation equal to λ . For a_1 one has

$$\lim_{\varepsilon \rightarrow 0} \omega_{\varepsilon \hat{n}}(a_1^n) = B_n$$

where B_n is the n -th Bell number, i.e., the number of partitions of a set of size n . The Bell numbers are the moments of the Poisson distribution with expectation equal to 1.

6. AN OPERATOR REPRESENTATION OF THE LIMITING CORRELATION FUNCTIONS

In the present section we explicitly realize the limiting correlation functions as correlation functions of certain operators acting in a suitable Hilbert space. Presence of delta functions in the limiting correlation functions suggests that they can be represented as correlation functions of certain white noise operators. Here such a representation is constructed in the special case using the results of⁷.

Let $g_0, g_1 \in \mathcal{H}$ satisfy the condition $\langle g_0, S_t g_1 \rangle = 0$ for any $t \in \mathbb{R}$. Define for $n, m = 0, 1$ the Hilbert space $\mathcal{K}_{nm} := L^2(\text{Spec } H_1, d\mu_{nm})$, where $\text{Spec } H_1 \subset \mathbb{R}$ is the spectrum of H_1 and $d\mu_{nm} := \langle g_n, P_E g_n \rangle \langle g_m, P_E \hat{n} g_m \rangle dE$. Let $\mathcal{K} := \bigoplus_{n,m=0,1} \mathcal{K}_{nm}$ and let

$\mathcal{H}_{WN} := \Gamma(L^2(\mathbb{R}, \mathcal{K}))$ be the symmetric Fock space over the Hilbert space of square integrable \mathcal{K} -valued functions on \mathbb{R} (abbreviation WN here stands for *White Noise*). Using the natural decomposition $\mathcal{H}_{WN} = \bigotimes_{n,m=0,1} \Gamma(L^2(\mathbb{R}, \mathcal{K}_{nm}))$ one can define the

creation and annihilation operator valued distributions $B_{m,n}^\pm(E, t)$ acting in \mathcal{H}_{WN} and satisfying the canonical commutation relations:

$$[B_{m,n}^-(E, t), B_{m',n'}^+(E', t')] = 2\pi \delta(t' - t) \delta(E' - E) \langle g_m, P_E g_{m'} \rangle \langle g_{n'}, P_E \hat{n} g_n \rangle \quad (18)$$

The operator valued distributions $B_{m,n}^\pm(E, t)$ are called **time-energy quantum white noise** due to the presence of $\delta(t' - t) \delta(E - E')$ in (18). Let define the number operators

$$\tilde{N}_{m,n}(E, t) := \sum_{n'=0,1} \frac{1}{\langle g_{n'}, P_E \hat{n} g_{n'} \rangle} B_{m,n'}^+(E, t) B_{n,n'}^-(E, t)$$

and denote $N_{g_m, g_n}(t) := \int dE [\tilde{N}_{m,n}(E, t) + B_{n,m}^-(E, t) + B_{m,n}^+(E, t)]$. Let $\Omega \in \mathcal{H}_{WN}$ be the vacuum vector.

Theorem 6 *Let $T_1 = |g_{m_1}\rangle \langle g_{n_1}|, \dots, T_k = |g_{m_k}\rangle \langle g_{n_k}|$, where $m_1, n_1, \dots, m_k, n_k \in \{0, 1\}$. One has the equality*

$$\lim_{\varepsilon \rightarrow 0} W_{\varepsilon, \hat{n}, T_1, 0, \dots, T_k, 0}(t_1, \dots, t_k) = \langle \Omega, N_{g_{m_1}, g_{n_1}}(t_1) \dots N_{g_{m_k}, g_{n_k}}(t_k) \Omega \rangle \quad (19)$$

Proof. r.h.s. of (19) has the form

$$\langle \Omega, N_{g_{m_1}, g_{n_1}}(t_1) \dots N_{g_{m_k}, g_{n_k}}(t_k) \Omega \rangle = \int dE_1 \dots dE_k \langle \Omega, [\tilde{N}_{m_1, n_1}(E_1, t_1) + B_{n_1, m_1}^-(E_1, t_1) + B_{m_1, n_1}^+(E_1, t_1)] \dots [\tilde{N}_{m_k, n_k}(E_k, t_k) + B_{n_k, m_k}^-(E_k, t_k) + B_{m_k, n_k}^+(E_k, t_k)] \Omega \rangle$$

Let us denote $\tilde{N}_{m,n}(t) := \int dE \tilde{N}_{m,n}(E, t)$. The truncated correlation function corresponds to the term

$$\int dE dE' \langle \Omega, B_{n_1, m_1}^-(E, t_1) \tilde{N}_{m_2, n_2}(t_2) \tilde{N}_{m_3, n_3}(t_3) \dots \tilde{N}_{m_{k-1}, n_{k-1}}(t_{k-1}) B_{m_k, n_k}^+(E', t_k) \Omega \rangle \quad (20)$$

Notice that $\tilde{N}_{m,n}(t)\Omega = 0$. Therefore (20) equals to

$$\int dE dE' \langle \Omega, [\dots [B_{n_1, m_1}^-(E, t_1), \tilde{N}_{m_2, n_2}(t_2)], \tilde{N}_{m_3, n_3}(t_3)] \dots \tilde{N}_{m_{k-1}, n_{k-1}}(t_{k-1})] B_{m_k, n_k}^+(E', t_k) \Omega \rangle$$

The commutators can be calculated by induction using the canonical commutation relations (18). The result is

$$(2\pi)^{k-2} \delta(t_2 - t_1) \dots \delta(t_{k-1} - t_{k-2}) \int dE dE' \langle g_{n_1}, P_E g_{m_2} \rangle \dots \langle g_{n_{k-2}}, P_E g_{m_{k-1}} \rangle \times \langle \Omega, B_{n_{k-1}, m_1}^-(E, t_{k-1}) B_{m_k, n_k}^+(E', t_k) \Omega \rangle \quad (21)$$

The last two-point correlation function can be calculated using the commutation relations (18). This gives for (21) the expression

$$(2\pi)^{k-1} \delta(t_2 - t_1) \dots \delta(t_k - t_{k-1}) \int dE \langle g_{n_1}, P_E g_{m_2} \rangle \dots \langle g_{n_{k-1}}, P_E g_{m_1} \rangle \langle g_{n_k}, P_E \hat{n} g_{m_1} \rangle$$

which coincides with the r.h.s. of (9) in the case $\omega_1 = \dots = \omega_k = 0$. \square

Remark 4 *The limiting correlation functions could be represented as expectations of certain quantum white noise operators in the general case if one could construct a Hilbert space \mathcal{H}_{WN} , a vector $\Omega \in \mathcal{H}_{WN}$, and operator valued distributions $B_{f,g}^\pm(E, t)$ and $\tilde{N}_{f,g}(E, t)$ in \mathcal{H}_{WN} with the property $B_{f,g}^-(E, t)\Omega = \tilde{N}_{f,g}(E, t)\Omega = 0$ and satisfying the commutation relations*

$$[B_{f,g}^-(E, t), B_{f',g'}^+(E', t')] = 2\pi \delta(t' - t) \delta(E' - E) \langle f, P_E f' \rangle \langle g', P_E \hat{n} g \rangle \quad (22)$$

$$[B_{f,g}^-(E, t), \tilde{N}_{f',g'}(E', t')] = 2\pi \delta(t' - t) \delta(E - E') \langle f, P_E f' \rangle B_{g',g}^-(E, t) \quad (23)$$

$$[\tilde{N}_{f,g}(E, t), \tilde{N}_{f',g'}(E', t')] = 2\pi \delta(t' - t) \delta(E' - E) [\langle g, P_E f' \rangle N_{f,g'}(E, t) - \langle g', P_E f \rangle N_{f',g}(E, t)] \quad (24)$$

Suppose there exist such operators. Define $N_{f,g}(t) := \int dE [\tilde{N}_{f,g}(E, t) + B_{g,f}^-(E, t) + B_{f,g}^+(E, t)]$. Then one can prove exactly in the same way as in Theorem 6 that

$$\lim_{\varepsilon \rightarrow 0} W_{\varepsilon, \hat{n}, |f_1\rangle \langle g_1|, 0, \dots, |f_n\rangle \langle g_n|, 0}(t_1, \dots, t_n) = \langle \Omega, N_{f_1, g_1}(t_1) \dots N_{f_n, g_n}(t_n) \Omega \rangle$$

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APPENDIX A

Here we make a connection between the objects defined in section II and the model of a test particle interacting with a dilute Bose gas (see Ref.⁹ for details).

The one particle Hilbert space for this model has the form $\mathcal{H} \equiv L^2(\mathbb{R}^3)$, where \mathbb{R}^3 is the 3-dimensional coordinate or momentum space. The one particle free evolution is a unitary group $S_t \equiv e^{itH_1}$ whose generator H_1 in the momentum representation is the multiplication operator by the function $\omega(\mathbf{k}) = |\mathbf{k}|^2/2m$, where m is the mass of a gas particle. The test particle is characterized by its Hilbert space \mathcal{H}_S and its free Hamiltonian H_S acting in \mathcal{H}_S which is assumed to have a discrete spectrum. The discrete set B is the set of all transition frequencies of the test particle, or equivalently, the spectrum of its free Liouvillean $-i[H_S, \cdot]$.

The dynamics of a test particle interacting with a gas is described by an evolution operator $U(t)$ acting in $\mathcal{H}_S \otimes \Gamma(\mathcal{H})$ and satisfying in the interaction picture, after the time rescaling $t \rightarrow t/\varepsilon$, the following Schrödinger equation

$$\frac{dU(t/\varepsilon)}{dt} = -i \left[\sum_{l,\omega} Q_{l,\omega} \otimes N_{T_l,\omega,\varepsilon}(t) \right] U(t/\varepsilon) \quad (25)$$

Here $Q_{l,\omega}$ are certain operators in \mathcal{H}_S such that $[H_S, Q_{l,\omega}] = -\omega Q_{l,\omega}$ and T_l are certain operators in \mathcal{H} . The explicit form of these operators is determined by the details of the microscopic interaction between the test particle and particles of the gas. Equation (25) is the place where the operators $N_{T_l,\omega,\varepsilon}(t)$ appear.

The condition $S_{-t}\hat{n}S_t = \hat{n}$ and positivity of \hat{n} imply that for this model \hat{n} is a multiplication operator by a function $n : \mathbb{R}^3 \rightarrow [0, \infty)$. The value $n(\mathbf{k})$ has the meaning of the density of gas particles at momentum \mathbf{k} . If the state of the gas is $\omega_{\varepsilon\hat{n}}$ then the density of gas particles and the rate of collisions between the test particle and the gas are of order ε . Thus the limit $\varepsilon \rightarrow 0$ is the low density limit. The limit is non-trivial since the dynamics is studied on the kinetic time scale of order $1/\varepsilon$.

APPENDIX B

Let $S(\mathbb{R})$ be the Schwartz space over \mathbb{R} and let $S'(\mathbb{R})$ be the dual space of distributions. We reproduce the following lemma from⁴.

Lemma 1 *One has the limit in $S'(\mathbb{R}) \times S'(\mathbb{R})$*

$$\lim_{\varepsilon \rightarrow 0} \frac{e^{itx/\varepsilon}}{\varepsilon} = 2\pi\delta(t)\delta(x)$$

Proof. Let $f, \phi \in S(\mathbb{R})$ and let \tilde{f} be Fourier transform of f , $\tilde{f}(\tau) = \int dx e^{i\tau x} f(x)$. One has the identities

$$I := \lim_{\varepsilon \rightarrow 0} \int dt dx \frac{e^{itx/\varepsilon}}{\varepsilon} f(x) \phi(t) = \lim_{\varepsilon \rightarrow 0} \int d\tau \phi(\varepsilon\tau) \int dx e^{i\tau x} f(x) = \lim_{\varepsilon \rightarrow 0} \int d\tau \phi(\varepsilon\tau) \tilde{f}(\tau)$$

Since $\tilde{f} \in S(\mathbb{R})$, the function $\phi(\varepsilon\tau)\tilde{f}(\tau)$ satisfies the conditions of the Lebesgue lemma which allows to exchange the limit and integration in the last expression. Thus

$$I = \phi(0) \int d\tau \tilde{f}(\tau) = 2\pi\phi(0)f(0) \quad \square$$

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